Graph spectral characterization of the XY model on complex networks

Sarah de Nigris, P. Expert, T. Takaguchi and R. Lambiotte

$^{\$}$ ENS, Lyon (FR), $^{*}$ Imperial College (UK), $^{^{a}}$ NICT, Tokyo (JP), $^{\£}$ UNamur, Namur (BE)
The XY model on networks can display different regimes:

Inspiration - Part II

These regimes are ignited by different network topologies

- regular
- small-world
- range constrained - Lace network
Our question

Moreover a kind of degeneracy emerges:

Is there a way to underpin the “important” structures for a given macroscopic state?
Outline

• The XY model on networks
  – Basic equations
  – Phenomenology recap
    • The Graph Signal Transform (GST)
      – What is it
      – How we use it
  • Results
  • Conclusions
The XY model: Basic Equations

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{J}{2k} \sum_{i,j} a_{i,j} (1 - \cos(\theta_i - \theta_j)) \]

\[ \ddot{\theta}_i = -\frac{J}{k} \sum a_{i,j} \sin(\theta_i - \theta_j) \]

\[ a_{i,j} = \begin{cases} 
1 & \text{if } i, j \text{ are connected} \\
0 & \text{otherwise} 
\end{cases} \]

\[ k = \frac{\sum_{i,j} a_{i,j}}{N} \]

\[ J > 0 \]
From the global side

**The XY model**

**Thermodynamical parameters:**

Energy

\[ H = E \Rightarrow \varepsilon = \frac{E}{N} \]

Temperature

\[ T = \frac{1}{N} \sum_i p_i^2 \]

**Order parameter: Magnetization**

\[
\begin{align*}
M &= \left\{ 
\begin{array}{l}
m_x = \frac{1}{N} \sum_i \cos \theta_i \\
m_y = \frac{1}{N} \sum_i \sin \theta_i
\end{array}
\right. \\
M &= |M| = \sqrt{m_x^2 + m_y^2}
\end{align*}
\]

\[ M \rightarrow 1 \]

\[ M \rightarrow 0 \]
Three global states (again):

- Non-magnetised
- Magnetised
- Supra-oscillant

Do they depend on the underlying network topology?
Yes, in some sense...

From this global view, let us go local again...
The graph signal transform in a nutshell

The XY model

The GST

Results

Conclusion

We choose a representation for the graph:

Eigenvalues and eigenvectors are a natural basis to represent a signal on the graph:

Graph signal:

\[ F_i(t) \quad i \in [0, N - 1] \]

Laplacian Matrix

\[ L_{i,j} = \overrightarrow{k}^T I - A_{i,j} \]

\[ \{v_\alpha, \lambda_\alpha\} \quad \alpha \in [0, N - 1] \]

\[ Lv_\alpha = \lambda_\alpha v_\alpha \]

Graph Signal Transform:

\[ \hat{F}_\alpha(t) = \sum_i F_i(t) v_i^\alpha \]


Sarah de Nigris
The GST decomposes the signal on nodes in "graph harmonics" like the Fourier Transform.

Our signal: \( \theta_i(t) \quad i \in [0, N - 1] \)  
\[ \Rightarrow \]  
\( \hat{\theta}_\alpha(t) \quad \alpha \in [0, N - 1] \)

Two focus questions:
1) Does each macrostate possess a definite fingerprint of selected harmonics?
2) How does this fingerprint change with graph topology?
Equilibrium Overview

First level of analysis: Macrostates → Stationary States

Average Power Spectrum

$$\frac{1}{T} \sum_{t \leq T} |\hat{\theta}_\alpha(t)|^2$$

Each macrostate has indeed a “harmonic fingerprint”
The only persistent mode is the one proportional to

\[ v_0 = \frac{1}{N} (1 \, 1 \, \ldots \, 1 \, 1) \]

Short-range correlation are at play

Weak selection of low eigenvalues modes
**The supra oscillant state:**
- Displays huge fluctuations
- Persistent over time
- Persistent with respect to the system size
- Exists on k-regular and range-constrained netw.

![Graph showing the supra oscillant state fingerprint](image)

Very strong excitation of a large interval of graph modes

“Stress” Tests

1) Topology invariant?

2) Size effects?

1) Topology invariant?

2) Size effects?

N=2048

N=4096

N=8192

Sarah de Nigris
Although the harmonic fingerprint of each collective state is the same, the network spectrum does change:
An important group of eigenvalues

Each macrostate selects from the underlying graph's spectrum a definite branch of eigenvalues

Ex: the supraoscillant state
Wrapping up

Some key points...

- Each macroscopic displays a definite harmonic fingerprint.
- In particular, the supra-oscillant state translates in a strong excitation of a large number of graph modes.
- These modes are localized in a specific branch of the graph spectrum.

S. d. N, P. Expert, T. Takaguchi, R. Lambiotte, arxiv:1611.01330

...and some (ongoing) perspectives...

- Reintroduce the time dimension: modes evolution.
- Get some insight on the network actual structures behind this selection in the eigenvectors space.
Sentiment Analysis in Large Social Networks

Tian Tong
Advisor: Rohit Negi

April 21, 2017
Outline

- **Motivation**
  - Majority sentiment detection in large social networks
  - Analyze whether detection is asymptotically accurate or not

- **Methods**
  - Asymptotic analysis using central limit theorem
  - Insights borrowed from statistical physics

- **Results**
  - Conditions on when detection error probability decays to 0
  - Relation to phase and phase transition phenomenon in statistical physics
  - Network examples & numerical experiments
Outline

- **Motivation**
  - Majority sentiment detection in large social networks
  - Analyze whether detection is asymptotically accurate or not

- **Methods**
  - Asymptotic analysis using central limit theorem
  - Insights borrowed from statistical physics

- **Results**
  - Conditions on when detection error probability decays to 0
  - Relation to phase and phase transition phenomenon in statistical physics
  - Network examples & numerical experiments
Classical binary detection problem

- Suppose binary $m \in \{-1, +1\}$ equi-probable
- Measure independently $n$ times over binary symmetric channel (BSC)
  - i.e., independent bit flips
  
  \[
  Y_i = \begin{cases} 
  -m & \text{w.p. } p \\
  m & \text{w.p. } 1 - p 
  \end{cases}
  \]
- Optimal detector $\hat{m}(Y) = \text{sign}(\sum_i Y_i)$
- Error probability of $\hat{m}$?
  
  \[
  P_e^{(n)} = P(\hat{m} \neq m) \leq ce^{-nE} \Rightarrow 0 \text{ as } n \to \infty
  \]
Majority sentiment detection problem

- Consider $n$ nodes (e.g., people in social network)
- Sentiment vector $X = (X_1, ..., X_n) \in \{-1, 1\}^n$
  - $\{X_i\}$ assumed independent, equi-probable

- Measurements $Y = (Y_1, ..., Y_n)$ over a BSC: $Y_i = \begin{cases} -X_i & \text{w.p. } p \\ X_i & \text{w.p. } 1 - p \end{cases}$
Majority sentiment detection problem

- Majority sentiment: \( m = \text{sign}(\sum_i X_i) \in \{-1, +1\} \) (\( n \) assumed odd)
- Majority sentiment detector: \( \hat{m} = \text{sign}(\sum_i Y_i) \)
- Examples: Sentiments from tweets; Polling for elections; Event detection using sensors, …
Example

Majority sentiment: $m = \text{sign}(\sum_i X_i) =$

Majority sentiment detector: $\hat{m} = \text{sign}(\sum_i Y_i) =$
Question: Behavior of detection error probability $P_e^{(n)} = P(\hat{m} \neq m)$?

Result: Asymptotic error probability

$$\lim_{n \to \infty} P_e^{(n)} = \frac{2}{\pi} \arcsin \sqrt{p}$$

Error probability never reduces to 0!

i.e., Even if every voter is polled, you cannot accurately predict election!
Sketch of proof: empty graph case

- Error probability:
  \[
P_e^{(n)} = P(m \neq \widehat{m}) = P(\text{sign}(\sum_i X_i) \neq \text{sign}(\sum_i Y_i)) = P(\sqrt{n} \overline{X_n} \sqrt{n} \overline{Y_n} < 0)
  \]

- Use central limit theorem:
  \[
  \left(\frac{\sqrt{n} \overline{X_n}}{\sqrt{n} \overline{Y_n}}\right) \xrightarrow{d} \left(\frac{X_{\lim}}{Y_{\lim}}\right) \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 - 2p \\ 1 - 2p & 1 \end{pmatrix}\right)
  \]

- By Berry-Esseen theorem, \(\lim_{n \to \infty} P_e^{(n)} = P(X_{\lim}Y_{\lim} < 0)\).

- Change of variables: \(U := \frac{X_{\lim} + Y_{\lim}}{2\sqrt{1-p}}, \ V := \frac{X_{\lim} - Y_{\lim}}{2\sqrt{p}}\), \(U, V \sim N(0,1)\) i.i.d.

  \[
  \lim_{n \to \infty} P_e^{(n)} = P\left(U^2 < \frac{p}{1-p} V^2\right) = \frac{2}{\pi} \arcsin \sqrt{p}
  \]
Simulation: empty graph case

\[ P_e(n) \]

\[ p = 0.2 \]
\[ p = 0.3 \]
\[ p = 0.4 \]
Sentiment vector $X = (X_1, ..., X_n) \in \{-1,1\}^n$ dependent through edges

- Ising Markov Random Field (MRF) prior $p(x) = \frac{1}{Z_n(\beta)} \exp(\beta \sum_{(i,j) \in E} x_i x_j)$

- $\beta > 0$ characterizes connection strength

- $\beta$ larger, neighbors are more probable to share the same sentiment

Decades ago, connection through telephone, $\beta$ is small
Nowadays, connection through online social network, $\beta$ is large

Sentiments of connected friends are more likely to be the same.
Majority sentiment detection in social network

- Measurements \( Y = (Y_1, ..., Y_n) \) over a BSC
- Majority sentiment \( m = \text{sign}(\sum_i X_i) \)
- Naïve detector \( \hat{m} = \text{sign}(\sum_i Y_i) \)
- Detection error probability \( P_e^{(n)} = P(\hat{m} \neq m) \)
Majority sentiment detection in social network

Question: Behavior of error probability $P_e^{(n)}$?

(Is it always bounded away from 0, as the empty graph case?)
Outline

- **Motivation**
  - Majority sentiment detection in large social networks
  - Analyze whether detection is asymptotically accurate or not

- **Methods**
  - Asymptotic analysis using central limit theorem
  - Insights borrowed from statistical physics

- **Results**
  - Conditions on when detection error probability decays to 0
  - Relation to phase and phase transition phenomenon in statistical physics
  - Network examples & numerical experiments
- Error probability $P_{e}^{(n)} = P(\sqrt{n}X_n\sqrt{n}Y_n < 0)\\$

- Analysis of $P_{e}^{(n)}$ can use conditional independence of measurements\\

- Conditional convergence of $\sqrt{n}Y_n$: for all $X$,\\
  $$\sqrt{n}(Y_n - (1 - 2p)\bar{X}_n)|X \xrightarrow{d} N(0, 4p(1 - p))$$\\

- Unconditional convergence of $\sqrt{n}Y_n$:\\
  $$\sqrt{n}(Y_n - (1 - 2p)\bar{X}_n) \xrightarrow{d} N(0, 4p(1 - p))$$\\

- Joint convergence of ($\sqrt{n}X_n, \sqrt{n}Y_n$): If $\sqrt{n}X_n \xrightarrow{d} \Phi$,\\
  $$\sqrt{n}(X_n, Y_n - (1 - 2p)\bar{X}_n) \xrightarrow{d} (\Phi, N(0, 4p(1 - p)))$$
Sketch of proof: conditional CLT

- Conditional independence of measurements: \( Y_i = \begin{cases} -X_i, & \text{w.p. } p \\ X_i, & \text{w.p. } 1 - p \end{cases} \)

- \( \{Y_i|X\} \) independent, but not identically distributed

\[
\text{Lindeberg CLT: } \{Z_i\} \text{ independent, mean zero, variance } \sigma_i^2, \text{ satisfies Lindeberg condition: } \frac{1}{s_n^2} \sum_{i=1}^{n} E \left(Z_i^2 I(|Z_i| \geq \epsilon s_n)\right) \to 0, \forall \epsilon > 0, \ s_n^2 = \sum_{i=1}^{n} \sigma_i^2,
\]

then: \( \frac{1}{s_n} \sum_{i=1}^{n} Z_i \overset{d}{\to} N(0,1) \).

- \( \{Y_i|X\} \) independent, mean \((1 - 2p)X_i\), variance \(4p(1 - p)\), satisfies Lindeberg condition, thus

\[
\sqrt{n}(\overline{Y_n} - (1 - 2p)\overline{X_n})|X \overset{d}{\to} N(0,4p(1 - p))
\]
Outline

- **Motivation**
  - Majority sentiment detection in large social networks
  - Analyze whether detection is asymptotically accurate or not

- **Methods**
  - Asymptotic analysis using central limit theorem
  - Insights borrowed from statistical physics

- **Results**
  - Conditions on when detection error probability decays to 0
  - Relation to phase and phase transition phenomenon in statistical physics
  - Network examples & numerical experiments
Main theory on asymptotic error probability

- Upper bound obtained by Hoeffding concentration inequality:

\[ P_e^{(n)} \leq E[\exp(-C_p (\sqrt{n X_n})^2)], \text{ where } C_p = \frac{(1-2p)^2}{8(1-p)^2} \]

- Asymptotic bounds obtained using central limit theorems:

\[ \liminf_{n \to \infty} P_e^{(n)} = \liminf_{n \to \infty} E \left[ Q\left( \frac{1 - 2p}{\sqrt{4p(1-p)}} |\sqrt{n X_n}| \right) \right] \]

\[ \limsup_{n \to \infty} P_e^{(n)} = \limsup_{n \to \infty} E \left[ Q\left( \frac{1 - 2p}{\sqrt{4p(1-p)}} |\sqrt{n X_n}| \right) \right] \]

- Closed-form limit obtained if \( \sqrt{n X_n} \xrightarrow{d} N(0, \sigma^2) \),

\[ \lim_{n \to \infty} P_e^{(n)} = \frac{1}{\pi} \arccot \left( \frac{1 - 2p}{\sqrt{4p(1-p)} \sigma} \right) > 0 \]

This is what happened in empty graph case
Main theory on asymptotic error probability

Question: Behavior of error probability $P_e^{(n)}$?

(decays to 0 or not?)

- Following result on $\sqrt{n X_n}$ is perhaps more intuitive:
  - **Case a)** If $\forall B > 0$, $\limsup_{n \to \infty} P(\sqrt{n X_n} \leq B) = 0$, then $\limsup_{n \to \infty} P_e^{(n)} = 0$
  - **Case b)** If $\exists B > 0$ s. t. $\liminf_{n \to \infty} P(\sqrt{n X_n} \leq B) > 0$, then $\liminf_{n \to \infty} P_e^{(n)} > 0$

![pdf](image)
Main theory on asymptotic error probability

Question: Behavior of error probability $P_e^{(n)}$?
   (decays to 0 or not?)

- Good news: limit of $P_e^{(n)}$ only depends on the concentration of $\sqrt{n X_n}$

- Bad news: but inference of the limit of dependent $\{X_i\}$ is very difficult!
Outline

- **Motivation**
  - Majority sentiment detection in large social networks
  - Analyze whether detection is asymptotically accurate or not

- **Methods**
  - Asymptotic analysis using central limit theorem
  - Insights borrowed from statistical physics

- **Results**
  - Conditions on when detection error probability decays to 0
  - Relation to phase and phase transition phenomenon in statistical physics
  - Network examples & numerical experiments
Phase in statistical physics

- Originally, Ising model appeared as a model for **ferromagnets**

\[ p(x) = \frac{1}{Z_n(\beta, h)} \exp \left( \beta \sum_{(i,j) \in E} x_i x_j + h \sum_{i \in V} x_i \right) \]

- \( x_i \): spins
- \( \beta \): inverse temperature
- \( h \): external (magnetic) field

- \( m(\beta, h) = \lim_{n \to \infty} E\overline{X}_n \) is called magnetization

- **Paramagnetic** phase: \( m(\beta, h) \to 0 \) as external field \( h \to 0 \)

- **Ferromagnetic** phase: \( m(\beta, h) \to \pm m^*(\beta) \) as external field \( h \to \pm 0 \)
  - \( m^*(\beta) \): residual magnetization left in ferromagnets after removing external magnetic field
Phase transition in statistical physics

- **Phase transition phenomenon** in statistical physics

  - High temperature (small $\beta$): iron is **paramagnetic** (non-magnet)
  - Low temperature (large $\beta$): iron is **ferromagnetic** (magnet)
Phase transition in statistical physics

- Some intuitions:
  - when external field becomes zero: \[ p(x) = \frac{1}{Z_n(\beta)} \exp(\beta \sum_{(i,j) \in E} x_i x_j) \]
  - under very small \( \beta \), \( \{X_i\} \) are nearly independent, \( \overline{X_n} \rightarrow 0 \)
  - under very large \( \beta \), \( \{X_i\} \) are likely as the same, \( \overline{X_n} \rightarrow \begin{cases} 1, \text{w.p. } 1/2 \\ -1, \text{w.p. } 1/2 \end{cases} \)

- paramagnetic
- ferromagnetic
Outline

- **Motivation**
  - Majority sentiment detection in large social networks
  - Analyze whether detection is asymptotically accurate or not

- **Methods**
  - Asymptotic analysis using central limit theorem
  - Insights borrowed from statistical physics

- **Results**
  - Conditions on when detection error probability decays to 0
  - Relation to *phase* and *phase transition phenomenon* in statistical physics
  - Network examples & numerical experiments
Under some technical conditions, e.g. correlation decay, the two cases for error probability $P_e(n)$ corresponds to the two phases:

- Ferromagnetic phase corresponds to case a) $\limsup_{n \to \infty} P_e(n) = 0$
- Paramagnetic phase corresponds to case b) $\liminf_{n \to \infty} P_e(n) > 0$
Motivation
- Majority sentiment detection in large social networks
- Analyze whether detection is asymptotically accurate or not

Methods
- Asymptotic analysis using central limit theorem
- Insights borrowed from statistical physics

Results
- Conditions on when detection error probability decays to 0
- Relation to phase and phase transition phenomenon in statistical physics
- Network examples & numerical experiments
Network examples: phases

Baxter “Exactly solved models in statistical mechanics.”

- **Empty Graph**: always in paramagnetic phase
- **Chain Graph**: always in paramagnetic phase
- **Complete Graph**: Curie-Weiss model scales down strength

\[
p(x) = \frac{1}{z_n(\beta)} \exp \left( \frac{\beta}{n} \sum_{i,j} x_i x_j \right)
\]

- If \( \beta < 1/2 \), in paramagnetic phase
- If \( \beta > 1/2 \), in ferromagnetic phase

- **Lattice Graph**:
  - If \( \beta < \frac{1}{2} \log(1 + \sqrt{2}) \), in paramagnetic phase
  - If \( \beta > \frac{1}{2} \log(1 + \sqrt{2}) \), in ferromagnetic phase
Network examples: error probability

- **Empty Graph**: 
  \[ \lim_{n \to \infty} P_e^{(n)} = \frac{2}{\pi} \arcsin \sqrt{p} > 0 \]

- **Chain Graph**: 
  \[ \lim_{n \to \infty} P_e^{(n)} = \frac{1}{\pi} \arccot \left( \frac{1-2p}{\sqrt{4p(1-p)}} e^\beta \right) > 0 \]

- **Complete Graph**?

- **Lattice Graph**?
Case a) If $\beta > 1/2$, using Hoeffding bound:

$$P_e^{(n)} \leq E[\exp(-C_p(\sqrt{nX_n})^2)]$$

$$\lim_{n \to \infty} P_e^{(n)} = 0, \quad \liminf_{n \to \infty} -\frac{1}{n} \log P_e^{(n)} > 0$$

- Error probability decays to 0 exponentially fast!
• Case b) If $\beta < 1/2$, then $\sqrt{nX_n} \xrightarrow{d} N\left(0, \frac{1}{1-2\beta}\right)$

$$\lim_{n \to \infty} P_e^{(n)} = \frac{1}{\pi} \arccot \left( \frac{1-2p}{\sqrt{4p(1-p)}} \frac{1}{\sqrt{1-2\beta}} \right) > 0$$
Lattice graph

- Calculations involve complicated techniques (cluster expansion).
- Experiments support the result:
  - For $\beta < \beta_c$, $\lim_{n \to \infty} P_e^{(n)} > 0$; for $\beta > \beta_c$, $\lim_{n \to \infty} P_e^{(n)} = 0$.

Fig. 3. Lattice graph: cdf of $X_n$ at $n = 961$.

Fig. 4. Lattice graph: $P_e^{(n)}$ v.s. $n$. 
Conclusion

- Majority sentiment detection with dependent sentiments
- Sentiment detection error probability behavior
- Relations to the phase and phase transition phenomenon
- In empty graph and chain graph, $P_e^{(n)}$ never reduces to 0.
- In complete graph and lattice graph, there exists a phase transition phenomenon, i.e., a critical connection strength:
  - above which, $P_e^{(n)}$ tends to 0
  - below which, $P_e^{(n)}$ remains bounded away from 0
- $P_e^{(n)}$ is strongly related to network connections.
  - Intuitively: If connections are strong, $P_e^{(n)}$ should tend to 0
Thermodynamic Limit of Interacting Agents over Time Re-wiring Sparse Random Networks

Augusto Santos
(CMU)
Jointly with: Soummya Kar (CMU), José M. F. Moura (CMU) and João Xavier (IST – University of Lisbon)
• Interacting Particle System:

\[ X^N(t) = \begin{bmatrix} X_1^N(t) & X_2^N(t) & \cdots & X_N^N(t) \end{bmatrix} \]

State of node 1
State of node 2
State of node N

• Thermodynamic Limit: \[ N \rightarrow \infty \]
Emergent Dynamics

- Large scale interacting simple agents output structured dynamics at the macroscale.

Flock of birds

Ant colonies

Many others:
- Epidemics
- Beehives
- Tumor growth

Seizures
Microscopic dynamics

Macroscale dynamics

Functional law of large numbers

\[
\left( \overline{Y}^N(t) \right) \xrightarrow{N \to \infty} (y(t))
\]
Interacting Particle Systems

Rest of the Network
Interacting Particle Systems

Rest of the Network

# neighbors $X$

$X$ # neighbors

# neighbors $X$
Interacting Particle Systems

Rest of the Network

\[ G(\text{blue, red}) = (\text{color}_1, \text{color}_2) \]
Interacting Particle Systems

Rest of the Network

\[ G(\text{red}, \text{blue}) = (\text{color}_1, \text{color}_2) \]
Interacting particle systems = clocks + G

\[ G \left( \text{red, blue} \right) = \left( \text{color}_1, \text{color}_2 \right) \]
Contagious Process
Contagious Process
Contagious Process
Functional weak law of large numbers (F.W.L.L.N)

\[ \left( \overline{Y}^N(t) \right) \xrightarrow{N \to \infty} \left( y(t) \right) \]
Functional weak law of large numbers (F.W.L.L.N)

\[
\left( \bar{Y}^N(t) \right) \xrightarrow{N \to \infty} (y(t))
\]

Action takes place

Few contact links

\[
\bar{Y}^N(t) \downarrow
\]

Many contact links

\[
\bar{Y}^N(t) \uparrow
\]
Easier vs Arbitrary Topology

Macrostates are tied to the microstate

Complete Network

Macrostates are Markov processes

Easier
\[ \bar{Y}^N(t) = \bar{Y}^N(0) + M^N(t) + \int_0^t F(X^N(s), A^N(s)) \, ds \]

- **Macroscopics**
- **Randomness**
- **Drift**

\[ X^N(t) = \begin{bmatrix} X_1^N(t) & X_2^N(t) & \cdots & X_N^N(t) \end{bmatrix} \]

\[ \uparrow \quad \uparrow \quad \uparrow \]

State of node 1  State of node 2  State of node N
\[
\overline{Y}^N(t) = \overline{Y}^N(0) + M^N(t) + \int_0^t F\left(\overline{Y}^N(s)\right) \, ds
\]

- **Macroscopics**
- **Randomness**
- **Drift**

**Complete Network**

\[
X^N(t) = \left[ X_1^N(t) \quad X_2^N(t) \quad \ldots \quad X_N^N(t) \right]
\]

- State of node 1
- State of node 2
- State of node N
Representation Theorem

\[ \bar{Y}^N(t) = \bar{Y}^N(0) + M^N(t) + \int_0^t F(X^N(s), A^N(s)) \, ds \]

**Macroscopics**
**Randomness**
**Drift**

**Arbitrary Topology**

\[ X^N(t) = \begin{bmatrix} X^N_1(t) & X^N_2(t) & \ldots & X^N_N(t) \end{bmatrix} \]

↑ State of node 1
↑ State of node 2
↑ State of node N
Randomly permute

$X^N(t-) = (1, 1, 0, 0, 0, 0, 0, 0, 0)$

Intermediate configuration

Randomly permute

$X^N(t) = (0, 1, 0, 0, 0, 1, 0, 1)$
Representation Theorem

\[ \bar{Y}^N(t) = \bar{Y}^N(0) + M^N(t) + \int_0^t F(X^N(s), A^N(s)) \, ds \]

- **Macroscopics**
- **Randomness**
- **Drift**

Arbitrary Topology

\[ X^N(t) = \begin{bmatrix} X^N_1(t) & X^N_2(t) & \ldots & X^N_N(t) \end{bmatrix} \]

State of node 1  State of node 2  State of node N
**Representation Theorem**

\[
\overline{Y}^N(t) = \overline{Y}^N(0) + M^N(t) + \int_0^t F\left(\overline{X}^N(s), \overline{A}^N(s)\right) ds
\]

- **Macroscopics**
- **Randomness**
- **Drift**

**Non-Markov Process**

\[
\left(\overline{Y}^N(t)\right) \xrightarrow{N \to \infty} \frac{d}{dt} y(t) = F(y(t))
\]

**ODE**
Theorem 9. Let $\bar{Y}^N(0) \Rightarrow y(0)$. We have

$\left( \bar{Y}^N(t) \right) \Rightarrow (y(t)) = (y_1(t), \ldots, y_k(t))$

where $(y(t))$ is the solution to the ODE

$$\dot{y}_k(t) = d \ y(t)^T \ (\Gamma \odot C(k)) \ y(t) \text{ for } k = \{1, 2, \ldots, K\}$$

(21)

with initial condition $y(0)$, $\Gamma = [\gamma_{m\ell}]_{m\ell}$, $C(k) = [c_{m\ell}(k)]_{m\ell}$ and $\odot$ is the pointwise Hadamard product.

Numerical Simulation

\[ N = \text{# of nodes per island} \]
Concluding Remarks and Future Direction

- Obtain macroscopic models from the natural microscopic rules of interactions;

- Valuable information about meta-stability;

- What is the class of network dynamics that leads to such concentration results.
$N \to \infty$

**Theorem 1:**

$$\left| F \left( X^N(t), A^N(t) \right) - f \left( Y^N(t) \right) \right| \xrightarrow{f.d.d.} 0$$

**Theorem 2:**

$$Y^N(t) = Y^N(0) + M^N(t) + \int_0^t F \left( X^N(s), A^N(s) \right) \, ds$$

\[\text{Weak convergence}\]

$$\dot{y}(t) = f(y(t)) = \gamma dy(t) (1 - y(t)) - \mu y(t)$$
**Non-Markov Process**

\[
\left( \mathbf{Y}^N(t) \right) \xrightarrow{N \to \infty} \frac{d}{dt} \mathbf{y}(t) = \mathbf{F}(\mathbf{y}(t))
\]

Macroscopic observable is tied to the high-dimensional microscopics

---

**Solution to ODE**

Macroscopic observable follow its own dynamics
Non-Markov Process

\[
\left( \bar{Y}^N(t) \right) \quad N \to \infty \quad \frac{d}{dt} y(t) = F(y(t))
\]

Macroscopic observable is tied to the high-dimensional microscopics

Macroscopic observable follow its own dynamics
Non-Markov Process

\[
\left( \overline{Y}^N(t) \right) \quad N \to \infty \quad \frac{d}{dt} y(t) = F(y(t))
\]

Macroscopic observable is tied to the high-dimensional microscopics

Macroscopic observable follow its own dynamics

Solution to ODE
Non-Markov Process

\[
\left( \bar{Y}^N(t) \right) \xrightarrow{N \to \infty} \frac{d}{dt}y(t) = F(y(t))
\]

Macroscopic observable is tied to the high-dimensional microscopics

Solution to ODE

Macroscopic observable follow its own dynamics
Numerical Simulation

(a) 100 nodes.
(b) 1000 nodes.
(c) 4000 nodes.

(a) 100 nodes.
(b) 1000 nodes.
(c) 4000 nodes.
\[ \dot{y}(t) = F(y(t)) \]

**VS**

\[
\begin{align*}
\dot{y}(t) &= F(y(t), x(t)) \\
\dot{x}(t) &= L(x(t))
\end{align*}
\]

\[ x(t) \in \mathbb{R}^{10^{23}} \]
1) Framework;
2) Some phenomena;
3) Framework again with prob. formulation;
4) Goal (show here the underlying cadlag-space and formalism etc);
5) Hydrodynamics and difficulty (high-dimension, etc.);
6) Glimpse on our proof;
7) Simulations
8) Conclusions (slow down the mixing, ) and relevance of the overall problem.
Dynkin's Formula
Dynkin's Formula
Theorem 9. Let $\mathbf{Y}^N(0) \Rightarrow \mathbf{y}(0)$. We have

$$
\left(\mathbf{Y}^N(t)\right) \Rightarrow (\mathbf{y}(t)) = (y_1(t), \ldots, y_k(t))
$$

where $(\mathbf{y}(t))$ is the solution to the ODE

$$
\dot{y}_k(t) = d \mathbf{y}(t)^\top (\Gamma \odot C(k)) \mathbf{y}(t) \quad \text{for } k = \{1, 2, \ldots, K\}
$$

(21)

with initial condition $\mathbf{y}(0)$, $\Gamma = [\gamma_{m\ell}]_{m\ell}$, $C(k) = [c_{m\ell}(k)]_{m\ell}$ and $\odot$ is the pointwise Hadamard product.
Some ideas:

1) Any particular example where an ODE is the natural model that emerges?

1) When the system is uncoupled it is easy, when it is completely coupled as well, but in-between things get involved.

2) Refer to the literature of hydrodynamics as emergent PDE’s.
Ingredients:

Sparse Interactions + Large scale = Challenging
Interactions + Large scale = tractable

Agent dynamics:

$N$ Agents

$\overline{Y}^N(t)$ Fraction of happy agents

Functional law of large numbers

$$\overline{Y}^N(t) \xrightarrow{N \rightarrow \infty} \frac{dy}{dt} = \gamma (1 - y(t)) + \mu y(t)$$
To give a glimpse on the proof or after presenting physically the model, refer that there is a representation of the process (representation theorem) and depict the cartoon of the proof: show that the rates converge and the martingale vanishes. Refer also that when the network is complete the rate reduces to a functional only in terms of $Y$ and that bla bla. But under a non-trivial sparsity, the term persists. What one would like to understand is whether such term converges to a functional in terms of $Y$. Our goal is to show that it is possible to obtain such result in an environment of sparse connection, under some fast-mixing dynamics.
Diffusion Inference for Oscillator Networks: Applications to Power System Dynamic Studies

Phuc Huynh and Hao Zhu
Dept. of Electrical & Computer Engineering
University of Illinois at Urbana-Champaign

Acknowledgements: NSF#1653706, Seibel Energy Inst., Q. Chen and A. Elbanna (CEE@UIUC)
Outline

- Motivation and context
- Oscillator network modeling
- Input-output response through modal analysis
- Synthetic and real data validation
- Concluding remarks
Motivation

- Many natural (even societal) networks have oscillatory dynamics
- Sensors prevail in real networks, providing huge volume of data during normal operating conditions (under small perturbations)
  - Phasor measurement unit (PMU) in power systems
  - Seismometers installed around the world
- Can one infer the propagation (diffusion) response to a disturbance input from the data characterizing normal operating behavior?
Electro-mechanical (EM) Wave in Power Grid
Related Work

- In seismology, estimating Green’s function of a medium to infer the propagation of earthquake waves
  - First for uncorrelated diffusive fields [Lobkis & Weaver’01] along with extensions in [Sneider’04, Wapenaar’04, Sneider et al’07]
  - Successfully validated on various types of seismic waves
  - Nonetheless, results limited to waves in continuum medium

- Internet tomography to infer the internal characteristics from indirect measurements [Coates et al’02]
  - Static inverse problem under network topological constraints

Our focus: establish an analytical framework to infer the propagation response to disturbances in (discrete) oscillatory networks
Oscillator Network

- Consider a network of \( n \) nodes, with node \( i \) coupled to the neighbors in set \( \mathcal{N}_i \).

- Second-order oscillatory dynamics
  \[ M_i \ddot{\delta}_i + D_i \dot{\delta}_i = u_i - \sum_{j \in \mathcal{N}_i} f_{ij} \]
  - \( \delta_i \) is the position/angle with speed/frequency \( \omega_i = \dot{\delta}_i \)
  - \( M_i \) (\( D_i \)) is inertia (damping) of oscillator \( i \)
  - \( u_i \) is the local input to oscillator \( i \)
  - \( f_{ij} \) represents the flow from oscillator \( i \) to \( j \)

- Based on Newton’s second law of motion
  - For power generators, rotational acceleration scales with net power
Linear Network Dynamics

- Under linearized systems, we have
  \[ f_{ij} = K_{ij}(\delta_i - \delta_j) \]
  - For lossless power system, the line flow follows
    \[ f_{ij} \propto \sin(\delta_i - \delta_j) \]

- Concatenating all variables into vectors/matrices:
  \[ M\ddot{\delta} + D\dot{\delta} + K\delta = u \quad \text{(ON)} \]
  - \( M \) and \( D \) are diagonal, while \( K \) is symmetric (graph Laplacian)

- State-space representation:
  \[
  \begin{bmatrix}
  \dot{\delta} \\
  \dot{\omega}
  \end{bmatrix} =
  \begin{bmatrix}
  0 & I \\
  -M^{-1}K & -M^{-1}D
  \end{bmatrix}
  \begin{bmatrix}
  \delta \\
  \omega
  \end{bmatrix} +
  \begin{bmatrix}
  0 \\
  M^{-1}u
  \end{bmatrix}
  \]
Goal: data-driven estimation of impulse response from any $u_k$ to $\omega_\ell$

$$T_{k\ell}(t) := \omega_\ell(t) \bigg|_{u=\delta(t)e_k}$$

Normal operating conditions with small perturbations

- In power systems, random variations of loads/generations

(as1) The system (ON) is excited by zero-mean white noise input

$$\mathbb{E}[u(t)] = 0,$$
$$\mathbb{E}[u(t)u^T(t-\tau)] = \Sigma \delta(\tau)$$

We will consider the (normalized) cross-correlation

$$C_{k\ell}(\tau) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \omega_k(t)\omega_\ell(t-\tau) d\tau$$
$$= \mathbb{E}[\omega_k(t)\omega_\ell(t-\tau)]$$
Modal Analysis

- To simply the modal analysis, consider uniform damping with \( D = \gamma M \)

\[
M\ddot{\delta} + D\dot{\delta} + K\delta = M\dot{\omega} + \gamma M\omega + K\delta = u \quad (\text{ON}')
\]

- Oscillation modes for (ON') related to a generalized eigen. problem

\[ KC = MCA \]

\[(as2) \text{ } M \text{ is positive definite (PD) and } K \text{ is symmetric}\]

**Lemma:** Under (as2), the eigenvectors in \( C \) are \( M \)-orthonormal; i.e.,

\[ C^TMC = I \]

with \( \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\} \) having eigenvalues of \( M^{-1/2}KM^{-1/2} \)
Decomposing into Uncoupled Modes

- Under the transformation $\delta = C z$, $(ON')$ becomes
  \[ \ddot{z} + \gamma \dot{z} + \Lambda z = v \]
  where the new input $v := C^T u$

- Each mode $(\ddot{z}_i + \gamma \dot{z}_i + \lambda_i z_i = v_i)$ is associated with eigenvalue pair
  \[ \alpha_i \pm j \beta_i = \frac{-\gamma \pm \sqrt{\gamma^2 - 4 \lambda_i}}{2} \]

- Under zero initialization
  \[ \dot{z}_i(t) = \int_0^\infty e^{-\frac{\gamma}{2} \tau} \cos(\beta_i \tau) v_i(t - \tau) d\tau \]
  \[ \omega_k(t) = \sum_{i=1}^n c_{ki} \dot{z}_i(t) = \int_0^\infty \left[ \sum_{i=1}^n c_{ki} e^{-\frac{\gamma}{2} \tau} \cos(\beta_i \tau) \right] v_i(t - \tau) d\tau \]

- IO response of interest given by
  \[ T_{k\ell}(\tau) = \sum_{i=1}^n c_{ki} c_{\ell i} e^{-\frac{\gamma}{2} \tau} \cos(\beta_i \tau) \]
Equivalence Results

\[ \omega_k(t) = \int_0^\infty \left[ \sum_{i=1}^n c_{ki} e^{-\frac{\gamma}{2} \tau} \cos(\beta_i \tau) \right] v_i(t - \tau) \, d\tau \]

\textbf{(as3)} Input noise variance proportional to nodal inertia; i.e., \( \Sigma = \mu I \)

- Each mode uniformly excited [cf. Lemma]

\[ \mathbb{E}[v(t)v^T(t - \tau)] = C^T \Sigma C \delta(\tau) = \mu I \delta(\tau) \]

\textbf{Prop:} Under (as1)-(as3), the IO response can be recovered by cross-correlating \( \omega_k \) and \( \omega_\ell \) as

\[ C_{k\ell}(\tau) \approx \mathbb{E}[\omega_k(t)\omega_\ell(t - \tau)] \]

\[ = \sum_{i=1}^n \mu c_{ki} c_{\ell i} e^{-\frac{\gamma}{2} \tau} \int_0^\infty e^{-\gamma \tau_2} \cos(\beta_i \tau_2) \cos(\beta_i (\tau_2 + \tau)) \, d\tau_2 \]

\[ \approx \frac{\mu}{2\gamma} \sum_{i=1}^n c_{ki} c_{\ell i} e^{-\frac{\gamma}{2} \tau} \cos(\beta_i \tau) \]
Toy Example

- A two-node system (IO response from \( u_2 \) to \( \omega_1 \))

\[
M = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}, \quad \gamma = 0.1, \quad \mu = 1
\]

\[\text{T}_2^1\]  \[\text{C}_2^1\]  \[\text{Normalized T}_2^1 \text{ vs C}_2^1\]

- Impulse response
- Cross-correlation
Small System Tests

- A small power system dynamics case with 3 generators (nodes)
- Synthetic data generated by randomly perturbing generator mechanical power input
- Linearized system model: matrix $K$ slightly asymmetric

\[
M = \begin{bmatrix}
47.28 & 0 & 0 \\
0 & 12.80 & 0 \\
0 & 0 & 6.02
\end{bmatrix}, \quad K = \begin{bmatrix}
0.777 & -0.223 & -0.547 \\
-0.280 & 1.189 & -0.909 \\
-0.578 & -0.901 & 1.479
\end{bmatrix}, \quad \gamma = 1
\]
Great match even with not perfectly symmetric model!
Nonlinear Model Outputs

- Generator control actions (turbine governor, exciter, and etc.)
- Good match in the peak locations, useful for determining the propagation speed of EM waves

![Graphs of nonlinear model outputs](image-url)
Real Data Tests

- Frequency measurements for the Eastern Interconnection (EI) system under normal grid operations
  - Collected from 10:00 AM to 10:15 AM on 01/20/2017 by FNET devices
- Compared to the propagation of an actual disturbance to the EI grid: the 2008 Florida blackout
2008 Blackout Replay

Florida Blackout Replay with FNET Data [Feb 26, 2008]
Time: 18:09:03.900 UTC  59.9984 Hz
Propagation Time Results

<table>
<thead>
<tr>
<th>Node #</th>
<th>Rec.</th>
<th>Est.</th>
</tr>
</thead>
<tbody>
<tr>
<td>601</td>
<td>1.5</td>
<td>1.2</td>
</tr>
<tr>
<td>671</td>
<td>0.7</td>
<td>0.5</td>
</tr>
<tr>
<td>682</td>
<td>2.6</td>
<td>2.6</td>
</tr>
<tr>
<td>726</td>
<td>2.5</td>
<td>2.6</td>
</tr>
<tr>
<td>729</td>
<td>2.3</td>
<td>2.6</td>
</tr>
<tr>
<td>756</td>
<td>1.6</td>
<td>1.9</td>
</tr>
<tr>
<td>767</td>
<td>1.5</td>
<td>1.9</td>
</tr>
<tr>
<td>781</td>
<td>2.1</td>
<td>2.1</td>
</tr>
<tr>
<td>787</td>
<td>1.6</td>
<td>1.6</td>
</tr>
<tr>
<td>823</td>
<td>1.5</td>
<td>1.7</td>
</tr>
</tbody>
</table>

Meter index
Recorded propagation time
Estimated propagation time
Conclusions

- Inferring the response to input disturbance is key to dynamics and stability analysis for oscillator networks

- We propose to **cross-correlate** the nodal measurements under small perturbations for estimating the propagation response
  - Stable systems with *uncoupled* modes
  - Each mode *equally excited* by white noise perturbations

- Successful validations on power grid using synthetic and real data

- Future work
  - Relaxed recovery conditions and/or bounded estimation error analysis
  - General inference tasks for oscillatory networks: sampling, interpolation, topology estimation
References


